Waves and Optics midterm exam 2025

12-12-2025 18:30-20:30

With answers

5 questions, 30 points

This 2-hour midterm exam tests your understanding of the main concepts of the Waves and Optics course. The last page of the exam is a formula sheet. Please write your name and student number on each page of your answer sheet and explain all your answers.

1. **Electromagnetic waves and anisotropic media** We are considering light transmission through a Magnesium Fluoride (MgF_2) crystal. This is a uniaxial crystal with a very high optical transmission from the UV to the IR. The indices of refraction as a function of wavelength are given in this table:

wavelength (nm)	n_o	n_e
5000	1.3340	1.3428
2500	1.3642	1.3752
1064	1.3731	1.3847
532	1.3789	1.3907
266	1.3989	1.4116

Table 1: The ordinary and extraordinary indices of refraction of MgF_2

(a) (2 points) Give the general wave equation for an electromagnetic wave propagating in a material like a MgF_2 (magnesium fluoride) crystal. Which terms do you include and which not, and why?
This is an anisotropic dielectric material. The material is polarized by the electromagnetic field of the light wave, and there is a spatial gradient of this polarization, so $\nabla \cdot \mathbf{P} \neq 0$*. It has no free charges (so the free current density* $\mathbf{J} = 0$*). Therefore the total wave equation is* $\nabla^2 \mathbf{E} - \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} = \mu_0 \frac{\partial^2 \mathbf{P}}{\partial t^2} - \frac{1}{\epsilon_0} \nabla(\nabla \cdot \mathbf{P})$.

(b) (2 points) We consider light entering the crystal at an angle $\theta = \pi/2$ with respect to the optical axis. Using the formula sheet, what is the index of refraction for 532 nm light that is linearly polarized along the $\hat{z}$-direction?
We can put $\theta = \pi/2$ *in the equation from the formula sheet for the extraordinary index of refraction. From this follows that* $n_e(\pi/2) = n_e$*. This is also how a uni-axial crystal is typically defined; the extraordinary index of refraction is oriented along the z-axis, while the order two* (n_x, n_y) *are equal. The answer to the question is therefore that* $n(\theta = \pi/2, \lambda = 532nm) = 1.3907$.

(c) (2 points) Use your understanding of the Lorenz model (which explains the frequency-dependent susceptibility in dielectrics) and the information given at the beginning of the

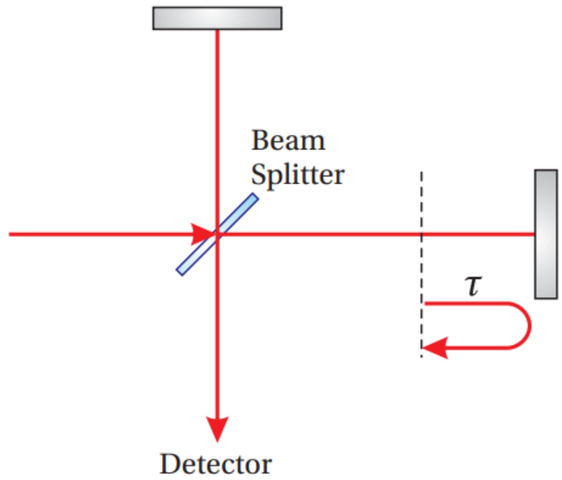


Figure 1: Michelson interferometer with a laser beam, a beam splitter two mirrors and a stepping mode that changes the pathlength to create constructive or destructive interference. The interference is recorded by a detector.

question to say whether MgF_2 has a resonance at really high frequencies or really low frequencies, compared to the visible part of the spectrum. Explain your answer.
The index of refraction is increasing a bit with increasing frequency (smaller wavelength); therefore the resonance frequency is at really high frequencies. The fact that the transmission is very high indicates that absorption is low; this indicated that we are not yet close to the resonance.

2. **Interferometers and coherence**.

   (a) (2 points) Explain the layout of all elements of a Michelson-Morley interferometer. Accompany your explanation with a sketch, pointing out all essential elements.
   A schematic of the Michelson interferometer is given in Figure 1. Incoming light is split in two parts, which travel along a vertical and horizontal arm. A delay can be introduced in one of the arms (horizontal in the sketch). The two parts of the wave are recombined on the beam splitter, and will interfere. Depending on their relative phase, and the degree of coherence, the light will interfere constructively towards the detector, or destructively back towards the light source.

   (b) (2 points) Explain what the coherence length is, and how such an interferometer can be used to determine the coherence length of a light source.
   The coherence length is the degree to which one part of a wave is similar to another part of the wave that is some length distance ahead (or behind). If the wave is highly coherent, an interference of the two parts of the wave that are separated by a large distance would still give a high contrast, because they would still have a well define phase difference. Since the MM interferometer does just this, it can be used - by introducing a time delay in one of the arms - to probe the coherence length. Say the coherence length is 1 cm, then a substantial reduction of contrast in the interference would occur when the difference in arm length is 1 cm. The exact definition of the amount of contrast reduction doesn't have to be given in the answer.

(c) (2 points) Give an example of a light source that produces light with a long coherence length, and give an example of a light source that produces light with a short coherence length. Explain why these light sources produce light with a high degree (or low degree) of coherence.

A light source which is practically monochromatic, like a laser, emits light with a large coherence length (up to 100s of kilometers); a source that has a wide spectrum, like the sun, produces light that has a very small coherence length (10s of micrometers).

(d) (2 points) A friend who is fascinated by gravitational waves asks your help to design an interferometer that would be sensitive to really small changes in the length difference of the two arms. Name two design choices you would make, and explain why they would help to make the interferometer more sensitive.

The sensitivity scales with the arm lengths; so increasing the arm length by placing the mirror further away is one design choice that could be taken. To increase the length further, one could place fabry-perot cavities in the arms; the light travels many times in the cavity before leaving, effectively increasing the arm length by a factor 100 to 1000. The sensitivity with which the interference pattern can be read out also scales with the intensity of the light; therefore, a brighter laser is a good design choice. Furthermore, especially if the interferometer has very large arms, it would be important to have a narrow linewidth laser that has a large coherence length. Another thing that would be good is to design the mounts of the mirrors such that they are immune to vibrations from the surroundings. Otherwise this is a noise source that would obscure a weak signal due to gravitational waves.

3. **Polarizers, polarization and index of refraction**. In this question we will consider the interaction of polarized light with various materials.

(a) (2 points) Let's start with just a light source followed by two rotatable polarizers and a screen. The light source emits a beam of unpolarized white light onto the first polarizer, which is set to transmit horizontally polarized light. You slowly rotate the second polarizer for a full rotation, and observe the screen. What is the intensity and color of the light that you expect to see on the screen, as you rotate the second polarizer? Explain your answer.

The intensity of the transmitted light is essentially $\cos^2\theta$, *with* θ *the angle between the two polarizers. So for a full rotation of the second polarizer we have two maxima, namely when the angle* $\theta = 0, \pi$. *This can be seen from the Jones matrix for a linear polarizer. If the polarizers are wavelength independent, which we can assume they are, then the colors doesn't change. In practice, when using plastic polarizer sheets, you typically see dark blue or purple colors leak through when the polarizers are perpendicular. You get a bonus point if you mention this last detail.*

(b) (2 points) Now we place a glass tube (about 1 meter long), filled with water, in between the two polarizers. You repeat the experiment above. What differences do you expect to see? Explain your answer.

The presence of the water doesn't really change anything, since it doesn't affect the polarization of the light. It might scatter the light a bit, so the intensity could be a bit reduced.

(c) (2 points) Now we dissolve a large amount of sugar into the water in the tube, and continue our experiments. We now rotate the first polarizer, while keeping the second polarizer fixed. What do you expect for the intensity, pattern and color of the light scattered in the tube (you could see this as you view the tube from the side), as you rotate this polarizer?

Explain your answer. Remember that sugar molecules are chiral and their interaction with light depends on the handedness (direction of circular polarization) of the light.
The sugar water has a different index of refraction for right-handed (rh) and left-handed (lh) circularly polarized light. Linear polarized light can be seen as a superposition of rh and lh light, where the relative phase of these two components determines the oscillation direction of the linear polarization. In addition to this, there is a frequency dependence of the index of refraction. These effects together result in a rotation of the linear polarization direction as the light propagates through the sugar water, with a color-dependent rotation rate. When viewed from the side, one therefore can see rotating bands of colors. The fact that these can be seen without polarizers can be understood from the dipole emission pattern; as for example the oscillation of electrons is caused by a vertically oscillating electric field, the emitted light is mostly directed in the directions perpendicular to the oscillation direction - so in the horizontal direction, for this example. These effects together leads to a spectacular effect that is called the Barbert pole effect.

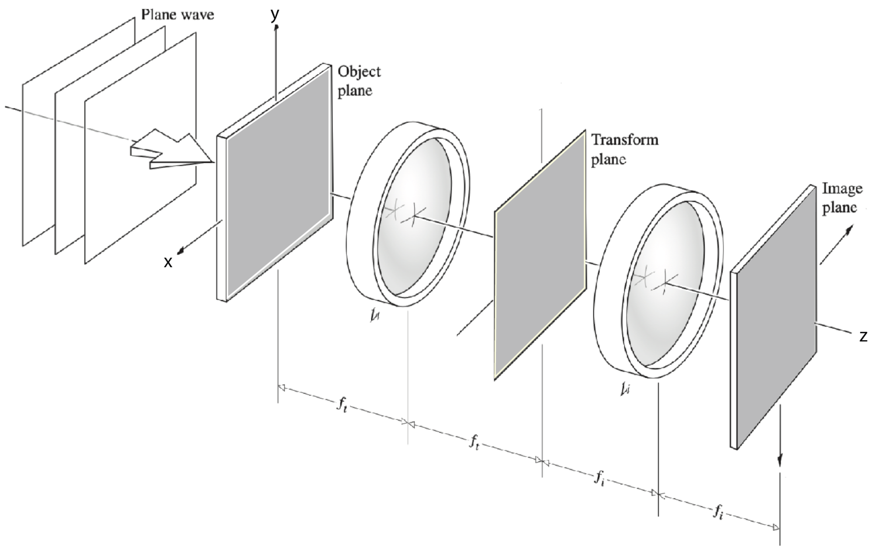
4. **Image formation.** In an optical system, the ABCD matrix relates the ray's initial point (y_1) and angle (θ_1) to its final point (y_2) and angle (θ_2):

$$\begin{bmatrix} y_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} y_1 \\ \theta_1 \end{bmatrix}.$$

(a) (2 points) Explain the physical meaning of A, B, C, and D in this context.
The ABCD matrix components represent how the initial ray position (y_1) and angle (θ_1) are transformed by an optical system into the final ray position (y_2) and angle (θ_2). A: ***Direct Point Transformation***
A describes how the initial ray position (y_1) contributes to the final ray position (y_2). - Example: In free-space propagation, $A = 1$, meaning the position along the axis remains unchanged.
B: ***Angle-to-Position Coupling*** *B determines how the initial ray angle (θ_1) influences the final ray position (y_2). - Example: In free-space propagation over a distance d, $B = d$, meaning a ray with a nonzero angle moves proportionally to the distance travelled.*
C: ***Position-to-Angle Coupling*** *C describes how the initial ray position (y_1) contributes to the final ray angle (θ_2). - Example: For a thin lens with focal length f, $C = -\frac{1}{f}$, indicating that the lens bends the ray more for larger ray positions.*
D: ***Direct Angle Transformation*** *D describes how the initial ray angle (θ_1) affects the final ray angle (θ_2). Example: In free-space propagation, $D = 1$, meaning the ray angle remains unchanged.*

(b) (2 points) Use appropriate ABCD matrices to show that rays that travel parallel to the optical axis and then pass through a thin lens cross that optical axis at the focal distance of the lens, regardless of their distance to the optical axis.
To answer this, you have to multiply the matrices for the three sections: flight some distance d to the thin lens (section I), the thin lens itself (section II), and then flight distance f (section III). So we get $\begin{bmatrix} 1 & f \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix} \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -y_1/f \end{bmatrix}$ *which shows that for any ray at distance y_1 parallel to the optical axis ($\theta_1 = 0$) we cross the optical axis ($y_2 = 0$) at the focal distance of the lens.*

5. **Diffraction** Imagine an electromagnetic plane wave, which encounters a transmission pattern at the location of the object plane in the image below. The diffracted light is collected by a lens. Downstream from the lens, in the transform plane a spatial filtering mask is placed. A second lens collects the light transmitted through this mask, and images that onto a screen at the image plane.



(a) (2 points) The object plane consists a large number (N) of closely-spaced vertical lines. They are specified to have a linewidth of w and a line spacing of h. In between the lines the object plane is transparent, and all light that hits the lines is blocked. The mask at the transform plane is such that it only transmits a small area in the center. What does the image in the image plane look like? Explain your answer.
The center part of the transform plane contains the dc-part (zero spatial frequency offset) of the object plane; so if only that part is transmitted, a uniform intensity will be visible in the image plane. The intensity will be a bit reduced, since part of the light (which contained the higher spatial frequency information) is blocked.

(b) (2 points) Let's consider just the plane wave and the object plane, and forget about the lenses (i.e. they are removed). We still consider the array of vertical lines in the object plane as specified before. The wavelength of incoming plane wave is adjusted with amount $\Delta\lambda$ from λ_1 to λ_2. Using the formula from the sheet, calculate the shift of the first order maxima of the diffraction pattern on a distant screen (a large distance z from the object plane) as the wavelength is adjusted.
Equations (11.28)-(11.30) in the BYU book on page 288 show how to solve this; essentially, the intensity formula peaks at specific conditions when $\frac{\pi h x}{\lambda z} = m\pi$, *where* m *is the order. From this we find that* $\Delta x = \frac{z}{h}\Delta\lambda$.

(c) (2 points) What parameter of the grating determines the width of the first order diffraction peak?
It is the number of slits, N*: as this is increased, the diffraction peaks get a reduced width. This is illustrated in figure 11.2 in the BYU book on page 287.*

Formula sheet

Electromagnetic wave equation in a medium

$\nabla^2\mathbf{E} - \mu_0\epsilon_0\frac{\partial^2\mathbf{E}}{\partial t^2} = \mu_0\frac{\partial\mathbf{J}_{\text{free}}}{\partial t} + \mu_0\frac{\partial^2\mathbf{P}}{\partial t^2} - \frac{1}{\epsilon_0}\nabla(\nabla\cdot\mathbf{P})$
Electric field of a decaying plane wave: $\mathbf{E}(r,t) = \mathbf{E_0}e^{-\frac{\kappa\omega}{c}\hat{\mathbf{u}}\cdot\mathbf{r}}\cos\left(\frac{n\omega}{c}\hat{\mathbf{u}}\cdot\mathbf{r} - \omega t + \phi\right)$

Jones matrices:

linear polarizer: $\begin{bmatrix}\cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta\end{bmatrix}$, half-wave plate: $\begin{bmatrix}\cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta\end{bmatrix}$,
quarter-wave plate: $\begin{bmatrix}\cos^2\theta + i\sin^2\theta & (1-i)\sin\theta\cos\theta \\ (1-i)\sin\theta\cos\theta & \sin^2\theta + i\cos^2\theta\end{bmatrix}$, Right circular polarizer: $\frac{1}{2}\begin{bmatrix}1 & i \\ -i & 1\end{bmatrix}$,
Left circular polarizer: $\frac{1}{2}\begin{bmatrix}1 & -i \\ i & 1\end{bmatrix}$.

Polarisation vector

$\begin{bmatrix}A \\ Be^{i\delta}\end{bmatrix}$, where $\mathbf{E}(z,t) = E_{\text{eff}}\left(A\hat{\mathbf{x}} + Be^{i\delta}\hat{\mathbf{y}}\right)e^{i(kz-\omega t)}$.
Left handed circularly polarized light: $\frac{1}{\sqrt{2}}\begin{bmatrix}1 \\ i\end{bmatrix}$, right handed circularly polarized light: $\frac{1}{\sqrt{2}}\begin{bmatrix}1 \\ -i\end{bmatrix}$

Lorentz model

$\ddot{\mathbf{r}}_e + \gamma\dot{\mathbf{r}}_e + \omega_0^2\mathbf{r}_e = \frac{q_e}{m_e}\mathbf{E}$, which leads to $\mathbf{P} = \epsilon_0\left(\frac{\omega_p^2}{\omega_0^2 - i\omega\gamma - \omega^2}\right)\mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$
and $(n+i\kappa)^2 = 1 + \chi(\omega) = 1 + \frac{\omega_p^2}{\omega_0^2 - i\omega\gamma - \omega^2}$

Crystals

Uniaxial crystal: $n_{x,y} = n_o$ and $n_e(\theta) = \frac{n_o n_e}{\sqrt{n_o^2\sin^2\theta + n_e^2\cos^2\theta}}$.

ABCD matrices

thin lens or mirror: $\begin{bmatrix}1 & 0 \\ -1/f & 1\end{bmatrix}$, with $1/f = (n-1)(1/R_1 - 1/R_2)$ for a thin lens, and $f = R/2$ for a mirror.
Propagating for a distance d: $\begin{bmatrix}1 & d \\ 0 & 1\end{bmatrix}$, Window, starting and stopping in air: $\begin{bmatrix}1 & d/n \\ 0 & 1\end{bmatrix}$.

Transmission grating

$I(x) = I_0\frac{\sin^2\left(\frac{\pi w}{\lambda z}x\right)}{(\frac{\pi w}{\lambda z}x)^2}\frac{\sin^2\left(N\frac{\pi h}{\lambda z}x\right)}{N^2\sin^2\left(\frac{\pi h}{\lambda z}x\right)}$, with w the slit width, N the number of slits, h the slit spacing, x the transverse distance in the screen plane, and z the distance from the grating to the screen.